Lecture Notes on Elliptic Systems of Phase Transition type

Nicholas Alikakos (University of Athens , EKPA)

Festum π Conference in Chania, 2024

1 One dimensional solutions: Heteroclinic Connections

(1) $$u'' - W_u(u) = 0 \quad , \quad u(\pm\infty) = a^{\pm} \;\; , \; a^+ \neq a^-$$

$a^{\pm} \in A = \{W = 0\}$, $\sharp A \geq 2$.

(2) $$J_{(s_1,s_2)}(u) := \int_{s_1}^{s_2} \left(\frac{1}{2}|u'|^2 + W(u)\right) dx \quad , \quad J(u) = J_{(-\infty,\infty)}(u)$$

Hypotheses

(H1) $W \in C^2$, $c_2|\xi|^2 \geq \xi^T \partial^2 W_u(a)\xi \geq c_1|\xi|^2$, $a \in A$

(H2) (a) $\liminf_{|u|\to+\infty} W(u) > 0$ or the weaker

(b) $\sqrt{W(u)} \geq \gamma(|u|)$, $\gamma : (0, +\infty) \to \mathbb{R}$, $\int_0^\infty \gamma(r)dr = +\infty$ or the stronger

(c) $W_u(u) \cdot u > 0$ if $|u| > M$.

Theorem 1.1. *(Existence) Under **(H1)**, **(H2)**(b), given $a^- \in A$, $\exists\, a^+ \in A \setminus \{a^-\}$ and u classical solution to*

(3) $$\frac{1}{2}|u'|^2 - W(u) = 0 \qquad (\textit{equipartition})$$

which minimizes J on

(4) $$\mathscr{A} = \left\{u \in W^{1,2}_{loc}(\mathbb{R};\mathbb{R}^m) \mid \lim_{x\to-\infty} u(x) = a^- \;\; , \; \lim_{x\to+\infty} u(x) \in A \setminus \{a^-\}\right\}$$

Note: (1) $W(u(x)) > 0$, $x \in \mathbb{R}$, by uniqueness for (3).
(2) If $|A|$ is even $\Rightarrow$ at least $\frac{|A|}{2}$ connections. If $|A|$ is odd $\Rightarrow$ at least $\frac{|A|+1}{2}$ connections.

Example 1.2. $m = 1$, $W(u) = \frac{1}{2}(u^2 - 1)^2$, bistable

$$u(x) = tanh x \;\; , \; a^{\pm} = \pm 1 \quad , \text{ unique modulo translations}$$

Remark 1.3. ***Exercise 1*** *($m = 2$, $g : (\mathbb{C}, W(z)dzd\overline{z}) \to$ Euclidean Plane isometry). Identify (u_1, u_2) with the complex number $z = u_1 + iu_2$ and write $W(u_1, u_2) = |f(z)|^2$. Assume $f' = g$ is holomorphic in D, open in $\mathbb{R}^2$. Let $u(x)$ a solution provided by Theorem 1.1.*

Then

$$Im\left(\frac{g(z) - g(a^-)}{g(a^+) - g(a^-)}\right) = 0 \ , \ for \ z \in \Gamma = \{u(x) \,|\, x \in \mathbb{R}\}$$

Moreover the set $g(\Gamma) = \{g(z) \,|\, z \in \Gamma\}$ is a line segment with end point $g(a^-)$, $g(a^+)$ and

$$\int_a^y \left(\frac{1}{2}|u'|^2 + W(u)\right) dx = \sqrt{2}\int_a^y |\frac{d}{dx}g(u)|dx = \sqrt{2}|g(u(y)) - g(a^-)|$$

Remark 1.4. *$m \geq 2$, Non uniqueness is possible*

Remark 1.5. *Sufficient condition for existence of connection $a_i \to a_j$, $\{W = 0\} = \{a_1, ..., a_N\}$*

Let

(5) $$\sigma_{ij} = \inf_{\mathscr{A}_{ij}} J(u) \ , \ \mathscr{A}_{ij} := \left\{u \in W^{1,2}_{loc}(\mathbb{R}; \mathbb{R}^m) \,|\, \lim_{x\to-\infty} u(x) = a_i \ , \ \lim_{x\to+\infty} u(x) = a_j\right\}$$

$i, j = 1, ..., N$. Then the condition

(6) $$\sigma_{ij} < \sigma_{ih} + \sigma_{hj} \ \ \forall a_h \in A \setminus \{a_i, a_j\} \quad \text{(Triangle Inequality)}$$

(6) is sufficient for the existence of an orbit connecting a_i to a_j.
(For $W(z) = |(z - z_1)(z - z_2)(z - z_3)|^2$ it is also necessary.)

Proof of Theorem 1.1.
1. Removing noncompactness due to translations
choose $r_0 > 0$ small such that

(7) $$\min_{a\in A \, , \, |u-a|<r_0} W(u) = W_0.$$

$\forall\ u(\cdot) \in \mathscr{A}\ ,\ \exists\ x_0$ such that $W(u(x_0)) = W_0$. Consider then the translates of $u \in \mathscr{A}$ with $W(u(0)) = W_0$. We restrict ourselves to this modified $\mathscr{A}$, which we still denote by $\mathscr{A}$.

2. $\underline{\mathscr{A} \neq \emptyset}$. Indeed given a^- consider the element $\tilde{a} \neq a^-$ in A closest to a^-.
Let

$$\tilde{u}(x) = (1 - (x + x_0))a^- + (x + x_0)\tilde{a}$$

Choose $x_0 \in (0, 1)$ s.t. $W(\tilde{u}(0)) = W_0$.
Let

$$J(\tilde{u}) = \sigma \tag{8}$$

3. $\underline{L^\infty \text{ bound}}$

$\exists\ M > 0$ depending on γ in **(H2)**(a) such that for $u \in \mathscr{A}$ with

$$J(u) \leq \sigma \Rightarrow ||u||_{L^\infty(\mathbb{R};\mathbb{R}^m)} \leq M. \tag{9}$$

Indeed let $u(\overline{x}) = M$, some $\overline{x}$

$$\sigma \geq J_{(-\infty,\overline{x})}(u) \geq \int_{-\infty}^{\overline{x}} \sqrt{2W(u(x))}|u'(x)|dx \geq \sqrt{2}\int_{|a^-|}^{M} \gamma(r)dr$$

4. Compactness

Let $\{u_j\} \subset \mathscr{A}$ minimizing sequence,

$$J(u_j) \to \inf_{\mathscr{A}} J(u) =: \sigma_0 \leq \sigma$$

and by (9),

$$|u_j(x_1) - u_j(x_2)| \leq |\int_{x_1}^{x_2} |u_j(\xi)|d\xi| \leq \sqrt{2}\sigma|x_1 - x_2|^{1/2} \tag{10}$$

By the Ascoli-Arzela theorem and a diagonal argument $\exists$ subsequence $\{u_j\}$,

$$u_j \to u^8\ \ ,\ \text{uniformly on compacts of}\ \mathbb{R}. \tag{11}$$

Note that $\{u_j\}$ is bounded in $W^{1,2}_{loc}(\mathbb{R};\mathbb{R}^m)$:

$$\frac{1}{2}\int_{l_1}^{l_2} |u_j'|^2 dx \leq J(u_j) \leq \sigma\ \ ,\ \int_{l_1}^{l_2} |u_j|^2 dx < C\ \ ,\ \text{by (9)}.$$

Hence

$$u_j \to u^* \text{ in } W^{1,2}_{loc}(\mathbb{R};\mathbb{R}^m) \tag{12}$$

$$\frac{1}{2}\int_{l_1}^{l_2} |u^*_x|^2 dx \leq \liminf_{j\to+\infty} \frac{1}{2}\int_{l_1}^{l_2} |u'_j|^2 dx \leq \liminf_{j\to+\infty} \frac{1}{2}\int_{\mathbb{R}} |u'_j|^2 dx \tag{13}$$

Also by Fatou,

$$\lim_{j\to+\infty} \int_{\mathbb{R}} W(u^*) \leq \liminf_{j\to+\infty} \int_{\mathbb{R}} W(u_j)dx \tag{14}$$

Thus

$$J(u^*) = \sigma_0 \tag{15}$$

5. <u>Boundary Conditions at $\pm\infty$</u>

By uniform continuity of u^* (10) and (15)

$$\lim_{x\to+\infty} W(u^*(x)) = 0 \tag{16}$$

If $\lim_{x\to+\infty} u^*(x)$ does not exist, then

$$\exists \ \{x_i^1\} \ , \ \{x_i^2\} \ \text{s.t.} \ u^*(x_i^1) \to a_1 \ , \ u^*(x_i^2) \to a_2$$

Hence $\exists \ \{\hat{x}_i\}$ s.t. $W(u^*(\hat{x}_i) \nrightarrow 0$, in contradiction to (16).
Thus

$$\lim_{x\to\pm\infty} u^*(x) = \begin{cases} a \\ a^+ \end{cases} \tag{17}$$

$a \ , \ a^+ \in A$.

<u>Claim 1:</u> $u^*(-\infty) = a^-$
Suppose for the sake of contradiction that $a \neq a^-$. Hence $\exists \ x_j \to -\infty$ s.t.

$$|u^*(x_j) - a| = \varepsilon_j \ , \ \varepsilon_j \to 0 \tag{18}$$

On the other hand $u_k(x) \to a^-$ as $x \to -\infty$.

By a diagonal argument we can choose subsequence of $\{u_j\}$ which we denote again by $\{u_j\}$ s.t.

$$|u_j(x_j) - a| \leq \varepsilon_j \;\;,\; j = 1, 2, ... \tag{19}$$

Since $u_j(x) \to a^-$ as $x \to -\infty$, it follows that

$$\int_{-\infty}^{x_j} \left(\frac{1}{2}|u_j'|^2 + W(u_j) \right) dx \geq \sigma_0 - \varepsilon_j \tag{20}$$

This is a consequence of $J(u_j) \geq \sigma_0$ and Exercise 2 below.

On the other hand from $W(u_j(0)) = W_0$ and the equicontinuity of u_j ((10)) for $\delta > 0$ small, fixed

$$\int_{-\delta}^{\delta} W(u_j) dx \geq \delta W_0 \tag{21}$$

Hence by (19), (20)

$$\int_{-\delta}^{\delta} \left(\frac{1}{2}|u_j'|^2 W(u_j) \right) dx \geq \sigma_0 - \varepsilon_j + \delta W_0 \tag{22}$$

contradicting that $\{u_j\}$ is a minimizing sequence.

Claim 2: $u^*(+\infty) = a^+ \neq a^-$

Once more we proceed by contradiction with a similar argument. So suppose $u^*(+\infty) = a^-$. As in (19), $\exists$ sequence $x_j \to +\infty \;\;,\; u_j(x_j) \to a^-$. From $W(u_j(0)) = W_0$ and the equicontinuity of u

$$\int_{-\delta}^{\delta} W(u_j) dx \geq \delta W_0 \tag{23}$$

for $\delta > 0$ small fixed.

Since $u_j \in \mathscr{A} \Rightarrow u_j(x) \to a \neq a^-$ as $x \to +\infty$.

Hence

$$\int_{x_j}^{+\infty} \left(\frac{1}{2}|u_j'|^2 + W(u_j) \right) dx \geq \sigma_0 - C_W \delta^2, \tag{24}$$

by Exercise 2.

Thus (23), (24) give

$$\int_{-\delta}^{+\infty} \left(\frac{1}{2}|u_j'|^2 + W(u_j) \right) dx \geq \sigma_0 - C_W \delta^2 + \delta W_0,$$

contradicting that $\{u_j\}$ is a minimizing sequence.

The proof of Theorem 1.1 is complete. □

Exercise 2: Let a_i , $a_j \in \{W = 0\}$, $s_+ > s_-$. Let $v : (s_-, s_+) \to \mathbb{R}^m$ minimize

$$J_{(s_-,s_+)}(v) = \int_{s_-}^{s_+} \left(\frac{1}{2}|v'|^2 + W(v)\right) dx$$

subject to $|v(s_-) - a_i| = |v(s_+) - a_j| = \delta$

Then

$$J_{(s^-,s^+)}(v) \geq \sigma_{ij} - C_W\delta^2 \quad , \; C_W \text{ a positive constant determined by } \textbf{(H1)}.$$

2 The Variational Maximum Principle

2.1 Hypotheses (W)

(25)
$$(\mathbf{H}_{VMP}) \begin{cases} W : \mathbb{R}^m \to \mathbb{R} \text{ , non-negative , } W \in C(\mathbb{R}^m; \mathbb{R}) \\ a \in \mathbb{R}^m \quad , \; W(a) = 0 \\ (0, r_0] \ni r \to W(a + r\xi) \text{ non decreasing , } \; W(a + r_0\xi) > 0 \end{cases}$$

$A \subset \mathbb{R}^n$, open, bounded, ∂A Lipschitz.

Theorem 2.1. *Let $u \in W^{1,2}_{loc}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$ be a minimizer of*

(26)
$$J(u, A) = \int_A \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dx$$

with respect to its Dirichlet conditions on ∂A:

(27)
$$J(u + v, A) \geq J(u, A) \quad , \; \forall\, v \in C^1_0(A; \mathbb{R}^m).$$

Assume that

(28)
$$|u(x) - r| \leq r \quad on \;\; A, \quad 0 < 2r \leq r_0$$

Then

(29)
$$|u(x) - a| \leq r \quad on \;\; A.$$

Comments (Difference with the usual maximum principle) This is a purely variational result while the usual maximum principle is a calculus fact that is based on the equation. We explain in terms of exercises:

Exercise 3: Consider for $W(u) = \frac{1}{2}(u^2-1)^2$,

$$\varepsilon^2 u'' - W'(u) = 0$$

For $\varepsilon > 0$ small there are (periodic) solutions which clearly violate (29).

Exercise 4: For W convex (29) holds by the classical maximum principle:

$$0 = u'' - W'(u) = u'' - \frac{W'(u)}{u} u$$

Remark (29) does not hold for local minimizers (small perturbations of u increase the energy).

Before giving the proof we introduce the polar representation for a map $u(x)$ that is a basic fact that will be utilized extensively in these notes.

(30)
$$u(x) = a + |u(x)-a| \frac{u(x)-a}{|u(x)-a|} =: a + \rho(x)\vec{n}(x)$$
$$\rho(x) := |u(x)-a| \ , \ \vec{n}(x) = \begin{cases} \frac{u(x)-a}{|u(x)-a|} & , \ u(x) \neq a \\ 0 & , \ u(x) = a \end{cases}$$

Formally

(31)
$$|\nabla u|^2 = |\nabla \rho|^2 + \rho^2(x)|\nabla \vec{n}(x)|^2$$

leading to the polar form of the energy

(32)
$$J_A(u) = \int_A \left\{ \frac{1}{2}\left(|\nabla\rho|^2 + \rho^2|\nabla\vec{n}(x)|^2\right) + W(a+\rho\vec{n}) \right\} dx$$

The point is that we will consider perturbations of $u(x)$ only in the radial point, keeping $\vec{n}$ fixed:

For $u(\cdot) \in W^{1,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$, and $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz, $f(0) = 0$ we consider the perturbation

$$\tilde{u}(x) = a + f(\rho(x))\vec{n}(x) \tag{33}$$

Then

$$\int_A |\nabla \tilde{u}|^2 dx = \int_A (|f'(\rho)\nabla\rho|^2 + f^2(\rho)|\nabla\vec{n}|^2) dx \tag{34}$$

It is a calculus fact for Sobolev functions that $\tilde{u}(\cdot) \in W^{1,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$ and (34), (32) hold rigorously.

Exercise 5: For variations of the special form

$$f(s) = sg(s) \ \ , \ g : \mathbb{R} \to \mathbb{R} \ \text{Lipschitz}$$

we can bypass $\vec{n}(x)$ and express $|\nabla\tilde{u}(x)|$ as follows:

$$|\nabla\tilde{u}|^2 = (f'(\rho))^2|\nabla\rho|^2 + (|\nabla u|^2 - |\nabla u|^2 - |\nabla\rho|^2)(\frac{f(\rho)}{\rho})^2. \tag{35}$$

Notice that $|\nabla u|^2 \geq |\nabla\rho|^2$, and that moreover if

$$|f'| \leq 1 \ \ , \ \ |g| \leq 1 \tag{36}$$

then

$$|\nabla\tilde{u}(x)|^2 \leq |\nabla u(x)|^2 \tag{37}$$

2.2 Proofs

Theorem 2.1 will follow from the following replacement result:

The cut-off lemma

Lemma 2.2. *Let W satisfy the hypotheses **(W)** and let $A \subset \mathbb{R}^n$, open bounded, with Lipschitz boundary. Suppose that*

$$u(\cdot) \in W^{1,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$$

If the following two condition hold
(I) $|u(x) - a| \leq r$ *on* ∂A , $0 < 2r \leq r_0$,
(II) $\mathcal{L}^n(A \cap \{|u(x) - a| > r\}) > 0$,
then $\exists\ \tilde{u}(\cdot) \in W^{1,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$ *such that*

$$\begin{cases} \tilde{u} = u \ \ on\ \ \partial A \\ |\tilde{u}(x) - a| \leq r \ \ , \ on\ \ A \\ J_A(\tilde{u}) < J_A(u) \end{cases}$$

Proof. We will assume A connected (no loss).

<u>Case 1</u>(easy) : <u>$\rho(x) \leq r_0$ a.e. in A</u>

Let

(38) $$f(s) = \frac{\min\{s, r\}}{s} s = g(s)s$$

By (37) ($|f'| \leq 1$, $|g| \leq 1$)

(39) $$\int_A |\nabla \tilde{u}(x)|^2 dx \leq \int_A |\nabla u(x)|^2 dx$$

Note that in the case of equality in (39), together with (35)

(40) $$\begin{aligned} 0 &= \int_A |\nabla \tilde{u}|^2 dx - \int_A |\nabla u|^2 dx \\ &= \int_A |\nabla \rho|^2((f'(\rho))^2 - 1)dx + \int_A (|\nabla u|^2 - |\nabla \rho|^2)(g^2(\rho) - 1)dx \\ &\leq - \int_{A \cap \{\rho \geq r\}} |\nabla \rho|^2 dx \end{aligned}$$

(41) $$\Rightarrow \nabla \rho = 0 \ \ , \ \text{a.e. on}\ \ A \cap \{\rho \geq r\}$$

Hence

(42) $$\nabla(\tilde{\rho} - \rho) = 0 \ \ , \ \text{a.e. on}\ \ A\ \ (\tilde{\rho} = f(\rho))$$

Hence

(43) $$\tilde{\rho}(x) - \rho(x) = \text{const.} \ \ , \ \text{a.e. in}\ \ A$$

and since

$$\begin{aligned} \tilde{\rho}(x) - \rho(x) = 0 \quad , \text{ on } \partial A \text{ in trace sense} \\ \Rightarrow \tilde{\rho}(x) - \rho(x) = 0 \ , \text{ a.e. in } A \end{aligned} \tag{44}$$

in contradiction to (II).

Thus we have strict inequality in (39).

On the other hand

$$\begin{aligned} \int_A W(\tilde{u}(x))dx &= \int_A W(a + f(\rho(x))\vec{n}(x))dx \\ &\leq \int_A W(a + \rho(x)\vec{n}(x))dx = \int_A W(u(x))dx \end{aligned} \tag{45}$$

Case 1 is settled.

<u>Case 2</u>: $\underline{\mathcal{L}^n(A \cap \{\rho > r_0\}) > 0}$

Consider the cut-off functions:

$$a(s) = \begin{cases} 1 & , \ s \leq r \\ \frac{2r-s}{r} & , \ r \leq s \leq 2r \\ 0 & , \ s \geq 2r \end{cases}$$

$$f(s) = \min\{s, r\}a(s) \ , \ g(s) = \frac{f(s)}{s}$$

(Reflection along $|u - a| = r$)

Define

$$\tilde{u}(x) = a + g(\rho(x))(u(x) - a) \tag{46}$$

By (37)

$$|\nabla \tilde{u}(x)|^2 \leq |\nabla u(x)|^2 \tag{47}$$

□

Lemma 2.3. *Let $A \subset \mathbb{R}^n$ open, bounded, connected with Lipschitz boundary, $f \in W^{1,2}(A;\mathbb{R})$ satisfies*

$$\begin{cases} f \leq \hat{r} \ \text{ on } \partial A \ \text{ (trace sense)} \\ \mathcal{L}^n(A \cap \{\hat{s} < f\}) > 0 \ , \text{ some } \hat{s} > \hat{r} \end{cases} \tag{48}$$

Then

$$\mathcal{L}^n(A \cap \{\hat{r} < f < \hat{s}) > 0 \tag{49}$$

Proof of Lemma 2.3. Let

$$E_1 = A \cap \{f \leq \hat{r}\} \ , \ E_2 = A \cap \{\hat{r} < f \leq \hat{s}\} \ , \ E_3 = A \cap \{\hat{s} < f\} \tag{50}$$

Define

$$\begin{aligned} \sigma(x) &= \min\{f(x), \hat{s}\} = \begin{cases} f(x) \ , & x \in E_1 \cup E_2 \\ \hat{s} \ , & x \in E_3 \end{cases} \\ \tau(x) &= \max\{\sigma(x), \hat{r}\} = \begin{cases} \hat{r} \ , & x \in E_1 \\ \hat{s} \ , & x \in E_3 \\ f(x) \ , & x \in E_2 \end{cases} \end{aligned} \tag{51}$$

Suppose for the sake of contradiction that

$$\mathcal{L}^n(A \cap \{\hat{r} < f \leq \hat{s}) = 0. \tag{52}$$

Then

$$\mathcal{L}^n(E_2) = 0 \quad \text{and} \quad \tau(x) = \begin{cases} \hat{r} \ , & x \in E_1 \\ \hat{s} \ , & x \in E_3 \end{cases}$$

$$\nabla \tau(x) = 0 \quad \text{a.e. in } A \tag{53}$$

On the other hand min and max of Sobolev functions produce Sobolev functions. Hence τ is Sobolev and the connectedness of A together with (53) implies that $\tau \equiv$ constant. Hence necessarily

$$\tau \equiv \hat{s} \ , \ \mathcal{L}^n(E_3) > 0$$

and also $\mathcal{L}^n(E_1) = 0$. Thus $f > \hat{s}$ a.e. in A. Thus $f \geq \hat{s}$ on ∂A, which contradicts $(48)_{(1)}$.

The proof of Lemma 2.3 is complete. □

3 The (Vector) Caffarelli-Córdoba Density Estimate

3.1 Introduction

In this lecture we are interested in entire solutions of

$$\Delta u - W_u(u) = 0 \quad , \quad u : \mathbb{R}^n \to \mathbb{R}^m \tag{54}$$

Hypotheses on W

$$\textbf{(H)} \begin{cases} W \in C^2(\mathbb{R}^m; [0\infty) \ , \ \{W = 0\} = \{a_1, ..., a_N\} \\ W_u(u) \cdot u > 0 \ \text{ if } \ |u| > M \\ c_2|\xi|^2 \geq \xi^T W_{uu}(a_i)\xi \geq c_1|\xi|^2 \ \ , \ i = 1, ..., N \end{cases}$$

Actually we will be interested in minimizing solutions:

Definition 3.1. *A function $u : \mathbb{R}^n \to \mathbb{R}^m$ is a minimizing solution of* (54) *in the sense of De Giorgi if*

$$J(u,\Omega) \leq J(u+v,\Omega) \ , \ \forall \ \text{bounded open set } \Omega \subset \mathbb{R}^n, \ \text{and } \forall \ v \in C^1_0(\Omega), \tag{55}$$
$$\text{where } \ J(u,\Omega) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + W(u) \right) dx$$

Exercise 6: Show that (54) is the Euler-Lagrange of J.

Remark 3.2. *We adopt this definition of minimizer as opposed to the standard*

$$\min \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla u|^2 + W(u) \right) dx$$

because it can be shown (Exercises 7 and 8) that for any solution of (54) *the following Liouville type estimates hold:*

$$\begin{cases} J(u, B_r(0)) = o(r^{n-2}) \ , \ \text{as } r \to +\infty, \ n \geq 3 \ \Rightarrow u \equiv \text{const.} \\ J(u, B_r(0)) = o(ln r) \ , \ \text{as } r \to +\infty, \ n = 2 \ \Rightarrow u \equiv \text{const.} \end{cases} \tag{56}$$

Hence for $n > 1$ any nontrivial solution of (54) *has the property that*

$$\int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla u|^2 + W(u) \right) dx = \infty$$

Exercise 7: (The Stress-Energy Tensor)

(i) Let $u : \mathbb{R}^n \to \mathbb{R}^m$

$$T_{ij}(u, \nabla u) = u_{x_i} \cdot u_{x_j} - \delta_{ij}(\frac{1}{2}|\nabla u|^2 + W(u)) \ \ i, j = 1, ..., n$$

Show that

$$divT = (\nabla u)^T \cdot (\Delta u - W_u(u))$$

(ii) Show that

$$trT = \frac{2-n}{2} \sum_{i=1}^{n} |u_{x_i}|^2 - nW(u) = -ng(u) + |\nabla u|^2$$

where $g(u) := \frac{1}{2}|\nabla u|^2 + W(u)$.

(iii) Show that

$$T + g(u)Id = (\nabla u)^T(\nabla u) \geq 0$$

Exercise 8: (Continuation)
Show that

(i)

$$\sum_{i,j} \int_{B_r} (x^i T_{ij})_{x_j} = -\int_{B_r} \left(\frac{n-2}{2} |\nabla u|^2 + nW(u) \right) \leq -(n-2) \int_{B_r} g(u)$$

(ii) By the divergence theorem

$$\sum_{i,j} \int_{B_r} (x^i T_{ij})_{x_j} dx = r \sum_{i,j} \int_{\partial B_r} T_{ij} \nu_i \nu_j = -r \int_{\partial B_r} \left(g(u) - |\frac{\partial u}{\partial \nu}|^2 \right)$$
$$\geq -r \int_{\partial B_r} g(u) = -r \frac{dJ_{B_r}(u)}{dr}$$

(iii) Combining (i),(ii) conclude that

$$-(n-2)J_{B_r}(u) \geq -r \frac{dJ_{B_r}(u)}{dr}$$
$$\Leftrightarrow \frac{d}{dr} \left(r^{-(n-2)} J_{B_r}(u) \right) \geq 0$$

(iv) Hence

$$J_{B_r}(u) \geq c r^{n-2}.$$

<u>**Exercise 9:**</u>
Combining the 1^{st} equality in (i) with the 2^{nd} in (ii), Exercise 8, derive Pohozaev's identity

$$\int_{B_r} \left(\frac{n-2}{2} |\nabla u|^2 + nW(u) \right) = r \int_{B_r} \left(\frac{1}{2} |\nabla u|^2 + W(u) - |\frac{\partial u}{\partial \nu}|^2 \right)$$

Remark 3.3. *The hypothesis **(H)**$_{(ii)}$ above implies easily the bounds*

(57) $$|u(x)| \leq M \ , \quad |\nabla u(x)| \leq \tilde{M} \ , \ \ x \in \mathbb{R}^n$$

Indeed if $|\{x \ : \ |u(x)| > M\}| \neq 0$ we take the truncation $v(x) = \dfrac{u(x)}{|u(x)|} M$, that clearly has less energy, hence contradicting that u is a miniizer. The gradient bound then follows from linear elliptic theory (Exercise).

3.2 The Basic Estimate

The following estimate indicates the "surface like" nature of minimizers.

Lemma 3.4. *Let $W : \mathbb{R}^m \to \mathbb{R}$ be continuous, $W \geq 0$ and assume that $\{W = 0\} \neq \varnothing$. Let u be minimizing (assume estimates (57)).*

Then there is a constant $\hat{C}_0 = \hat{C}_0(W, M, \tilde{M})$, independent of x_0 and such that

(58) $$B_r(x_0) \subset \Omega \Rightarrow J(u, B_r(x_0)) \leq \hat{C}_0 r^{n-1}, \quad r > 0$$

Proof. Note $g(u) = \frac{1}{2} |\nabla u|^2 + W(u)$ is bounded in Ω and it follows

$$J(u, B_r(x_0)) \leq C_1 r^n \leq C_1 r^{n-1} \text{ for } n \leq 1$$

For $r > 1$ define the competitor

$$v(x) = \begin{cases} a, & |x - x_0| \leq r - 1 \\ (r - |x - x_0|)a + (|x - x_0| - r + 1)u(x), & r - 1 < |x - x_0| \leq r \\ u(x), & |x - x_0| > r \end{cases} \tag{59}$$

The definition and the minimizing property of u over balls imply

$$J(u, B_r(x_0) \leq J(v, B_r(x_0)) = J(v, B_r(x_0) \setminus B_{r-1}(x_0)) \leq C_2 r_0^{n-1}$$

□

3.3 Motivation

We now introduce the density estimate by considering first the motivation behind it that comes from the sharp inference case of minimal surfaces or better minimal partitions. Our argument below is formal.

Consider a minimal surface $\Sigma^{n-1} \subset \mathbb{R}^n$. Let $x \in \Sigma^{n-1}$ and consider $B_r(x)$ which is partitioned by Σ^{n-1} in two parts D_r and D_r^c.

Let

$$V(x) = \mathcal{L}^n(D_r), \quad A(r) = \mathcal{H}^{n-1}(\Sigma^{n-1} \cap B_r) \tag{60}$$

S_r stands for the spherical cup bounding D_r, $\mathcal{H}^{n-1}$ for the $(n-1)-$ Hausdorff measure.

Consider the following computation.

$$\begin{aligned} V(r) &\leq C[\mathcal{H}^{n-1}(\Sigma^{n-1} \cap B_r + \mathcal{H}^{n-1}(S_r)]^{\frac{n}{n-1}}, \quad \text{by the isometric ineq.} \\ &\leq C[2\mathcal{H}^{n-1}(S_r)]^{\frac{n}{n-1}}, \quad \text{by minimality of } \Sigma^{n-1} \\ &\leq C[V'(r)]^{\frac{n}{n-1}}, \quad \text{by the coarea formula or Fubini} \end{aligned} \tag{61}$$

From (58) it follows that if $\mu_0 = V(r_0) > 0$, some $\delta > 0$, then

$$V(r) \geq Cr^n, \quad r \geq r_0, \quad C = C(n) \tag{62}$$

The analogy with the diffuse interface is via the identification

$$A(r) = \int_{B_r \cap \{|u-\alpha| \leq \lambda\}} W(u)dx, \quad V(r) = \mathcal{L}^n(B_r \cap \{|u-\alpha| > \lambda\}) \tag{63}$$

where $W(a) = 0$ and $\lambda > 0$ any number such that

$$d_0 = \text{dist}(\alpha, \{W = 0\} \setminus \{\alpha\}) \geq \lambda. \tag{64}$$

3.4 Hypothesis for the Density Estimate

$$(H_d) \begin{cases} W \in C(\mathbb{R}^m, [0, \infty)), \quad W(\alpha) = 0 \\ (i)\ \underline{0 < \alpha < 2:} \, W \text{ differentiable in a deleted neighborhood of } a \\ \quad \dfrac{d}{dp} W(a + \rho\xi) \geq \alpha\, c^* \rho^{\alpha-1}, \quad \rho \in [0, \rho_0], \quad \forall \xi \ : |\xi| = 1 \\ (ii)\ \underline{\alpha = 2:} \, W \text{ is } C^2 \text{ in a neighborhood of } \ a \\ \quad C_2 |\xi|^2 \geq \xi^T W_{uu}(a) \xi \geq C_1 |\xi|^2 \end{cases}$$

Theorem 3.5. *Assume W satisfies (H_d), Ω open, $n \geq 1$, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ minimizing. Then for any $\mu_0 > 0$ and any $\lambda \in (0, d_0)$ the condition*

$$\mathcal{L}^n(B_{r_0}(x_0) \cap \{|u - \alpha| > \lambda\} \geq \mu_0 \tag{65}$$

implies

$$\mathcal{L}^n(B_r(x_0) \cap \{|u - \alpha| > \lambda\}) \geq Cr^n, \quad r \geq r_0 \tag{66}$$

as long as $B_r(x_0) \subset \Omega$, $C = C(W, \mu_0, \lambda, r_0, M, \tilde{M})$.

Exercise 10: Utilizing Lemma 3.4 show that the validity of the theorem for one value of $\lambda \in (0, d_0)$ implies its validity for all $\lambda' \in (0, d_0)$.

Exercise 11: Assume $\{W = 0\} = \{a_1, a_2\}$, and assume that (H_d) holds. Then given $0 < \theta < |\alpha_1 - \alpha_2|$, the condition

$$\mathcal{L}^n(B_1(x_0) \cap \{|u - a_1| \leq \theta\}) \geq \mu_0 > 0 \tag{67}$$

implies the estimate:

$$\mathcal{L}^n(B_r(x_0) \cap \{|u - a_1| \leq \theta\}) \geq Cr^n \tag{68}$$

for $r \geq 1$ as long as $B_r \subset \Omega$.

Proof. **STEP I** : The Identity

We recall the polar form introduced in (30). For $u(\cdot) \in W^{1,2}(B_r; \mathbb{R}^m) \cap L^\infty(B_r; \mathbb{R}^m)$, $B_r = \{|x - x_0| < r\}$,

$$u(x) = a + q^u(x)\vec{n}^u(x)$$

where

$$q^u(x) = |u(x) - a|, \quad \vec{n}^u(x) = \tfrac{u(x)-a}{|u(x)-a|}, \quad \text{if } u(x) \neq a$$
$$q^u \in W^{1,2}(B_r) \cap L^\infty(B_r), \quad \nabla\vec{n}^u \text{ measurable}, \quad q^u|\nabla\vec{n}^u| \in L^2(B_r)$$

$$\int_{B_r} |\nabla u|^2 \, dx = \int_{B_r} |\nabla q^u|^2 \, dx + \int_{B_r} (q^u)^2 |\nabla\vec{n}^u|^2 \, dx \tag{69}$$

As in section 2 we consider variations where only the radial part q^u is modified while $\vec{n}^u$ is kept fixed:

$$h = a + q^h \vec{n}^u(x), \quad \sigma = a + q^\sigma \vec{n}^u(x) \tag{70}$$

$$q^\sigma = min\{q^h, q^u\} \tag{71}$$

where

$$q^h \in W^{1,2}(B_r) \cap L^\infty(B_r), \quad q^h \geq 0 \tag{72}$$

with a suitable radial C^1 map, with

$$q^h \geq q^u \quad \text{on} \quad \partial B_r \tag{73}$$

(69) with $u = \sigma$ yields

$$\int_{B_r} |\nabla \sigma|^2 \, dx = \int_{B_r} |\nabla q^\sigma|^2 \, dx + \int_{B_r} (q^\sigma)^2 |\nabla \vec{n}^u|^2 \, dx \tag{74}$$

Thus we derive the identity:

$$\begin{aligned} &\frac{1}{2}\int_{B_r} (|\nabla q^u|^2 - |\nabla q^\sigma|^2) \, dx = \\ J_{B_r}(u) - J_{B_r}(\sigma) + \frac{1}{2}\int_{B_r} &((q^\sigma)^2 - (q^u)^2)|\nabla \vec{n}^u|^2 \, dx + \int_{B_r} (W(\sigma) - W(u)) \, dx \\ &\leq \int_{B_r} (W(\sigma) - W(u)) \, dx \end{aligned} \tag{75}$$

where in deriving the last inequality we used that $q^\sigma \leq q^u$ and the minimizing property of u. Notice the similarity of (75) with (61).

STEP II: The isoperimetric estimate

Recall the Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |f|^{\frac{n-1}{n}} dx\right)^{\frac{n-1}{n}} \leq C(n) \int_{\mathbb{R}^n} |\nabla f| dx, \quad \forall f \in W^{1,2}(\mathbb{R}^n), \quad n \geq 2 \tag{76}$$

with ρ_0 as in (H_d), we define the cut-off

$$\beta = min\{q^u - q^\sigma, \lambda\} \text{ in } B_r, \text{ with } \lambda > 0, \text{ small } \lambda \leq \rho_0$$

and apply (76) to β^2

$$\begin{aligned} \left(\int_{B_r} \beta^{\frac{2n}{n-1}} dx\right)^{\frac{n-1}{n}} &= \left(\int_{B_r} (\beta^2)^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \\ \leq C(n) \int_{B_r} |\nabla(\beta^2)| dx &\leq 2C(n) \int_{B_r \cap \{q^u - q^\sigma \leq \lambda\}} |\nabla \beta||\beta| dx \end{aligned} \tag{77}$$

where we have utilized that $\beta = 0$ on ∂B_r and $\nabla \beta = 0$ a.e on $q^u - q^\sigma > \lambda$.

By Young

$$\begin{aligned} \left(\int_{B_r} \beta^{\frac{2n}{n-1}} dx\right)^{\frac{n-1}{n}} &\leq 2C(n) \int_{B_r \cap \{q^u - q^\sigma \leq \lambda\}} |\nabla \beta||\beta| dx \\ \leq C(n) A \int_{B_r \cap \{q^u - q^\sigma \leq \lambda\}} |\nabla \beta|^2 dx &+ \frac{C(u)}{A} \int_{B_r \cap \{q^u - q^\sigma \leq \lambda\}} \beta^2 dx \\ \leq C(n) A \int_{B_r} |\nabla (q^u - q^\sigma)|^2 dx &+ \frac{C(u)}{A} \int_{B_r \cap \{q^u - q^\sigma \leq \lambda\}} (q^u - q^\sigma)^2 dx \end{aligned} \tag{78}$$

Noting the identity

$$|\nabla(q^u - q^\sigma)|^2 = |\nabla q^u|^2 - |\nabla q^\sigma|^2 - 2\nabla q^\sigma (\nabla q^u - \nabla q^\sigma) \tag{79}$$

we can bound the right - hand side of (78) utilizing the identity (75).

$$\begin{aligned} \left(\int_{B_r} \beta^{\frac{2n}{n-1}} dx\right)^{\frac{n-1}{n}} &\leq 2C(n) A \left(\int_{B_r} (W(\sigma) - W(u)) dx - \int_{B_r} \nabla q^\sigma (\nabla q^u - \nabla q^\sigma) dx\right) \\ &+ \frac{C(u)}{A} \int_{B_r \cap \{q^u - q^\sigma \leq \lambda\}} (q^u - q^\sigma)^2 dx \end{aligned} \tag{80}$$

Assuming that $q^h \in W^{1,2}(B_r) \cap L^\infty(B_r)$ can be chosen so that

(81) $$q^h = 0 \quad \text{on} \quad B_{r-T}, \quad \text{some fixed} \quad T > 0$$

and this

$$q^\sigma \text{ on } B_{r-T} \Leftrightarrow \sigma = a \text{ on } B_{r-T}$$

we can estimate

$$\left(\int_{B_r} \beta^{\frac{2n}{n-1}} dx\right)^{\frac{n-1}{n}} \geq \left(\int_{B_{r-T} \cap \{q^u > \lambda\}} \beta^{\frac{2n}{n-1}} dx\right)^{\frac{n-1}{n}} \geq \lambda^2 \mathcal{L}^n(B_{r-T} \cap \{q^u > \lambda\})^{\frac{n-1}{n}}$$

and obtain from (80) that

(82) $$\begin{aligned} &\lambda^2 \mathcal{L}^n(B_{r-T} \cap \{q^u > \lambda\})^{\frac{n-1}{n}} \\ &\leq 2C(n)A\left(\int_{B_r} (W(\sigma) - W(u))dx - \int_{B_r} \nabla q^\sigma(\nabla q^u - \nabla q^\sigma)dx\right) \\ &\quad + \frac{C(n)}{A} \int_{B_r \cap \{q^u - q^\sigma \leq \lambda\}} (q^u - q^\sigma)^2 dx \end{aligned}$$

Exercise 12: $(\alpha < 2)$

Consider the O.D.E

$$\begin{cases} q' = \frac{2}{2-\alpha} C^{\frac{\alpha}{2}} q^{\frac{\alpha}{2}}, \quad q'' = \frac{2\alpha}{(2-\alpha)^2} C^\alpha q^{\alpha-1} \\ q(0) = 0 \end{cases}$$

Show that it has the family of nontrivial solutions

$$q(s) = \begin{cases} C^{\frac{\alpha}{2-\alpha}} s^{\frac{2}{2-\alpha}}, \quad s > 0 \\ 0, \quad s \leq 0 \end{cases}$$

Let $\tilde{q}(s) = Cs^{\frac{2}{2-\tau}}$, $\tilde{q}(s) = 0$ for $s \leq 0$, $\tau = max\{\alpha, 1\}$

Show that $q^h(x) = \tilde{q}(|x| - (r - T))$ is Sobolev.

Notice that $q''(0)$ is finite, $q'(0) = 0$.

STEP III: The case $0 < \alpha < 2$

A. We recall (H_d)(i), which is modeled after $W(u) \sim |u-a|^2$ for $u \sim a$. The validity of (57) needs some attention here and will be discussed later.

We estimate the first-hand side of (82).

We begin with B_{r-T}. Since $q^\sigma = 0$ on B_{r-T}, the first-hand side reduces to.

$$\begin{aligned} I &= -2C(n)A\int_{B_{r-T}} W(u)dx + \frac{C(n)}{A}\int_{B_{r-T}\cap\{q^u\le\lambda\}} (q^u)^2 dx \\ &\le -2C(n)A\int_{B_{r-T}\cap\{q^u\le\lambda\}} W(u)dx + \frac{C(n)}{A}\int_{B_{r-T}\cap\{q^u\le\lambda\}} (q^u)^2 dx \end{aligned} \tag{83}$$

<u>**Exercise 13:**</u>

Assume $\lambda \le \rho_0$, ρ_0 as on (H_d). Then there exists $A_0 > 0$ independent of r, such that

$$I \le -\frac{C(n)}{2}A\int_{B_{r-T}\cap\{q^u\le\lambda\}} W(u)dx, \quad \text{for} \quad A > A_0 = \sqrt{2\lambda^{2-\alpha}/3C^*} \tag{84}$$

(Utilize lower bound in (H_d)(i)).

B. Next we consider the right-hand side of (82) on $B_r \setminus B_{r-T}$

Set

$$I_1 = 2C(n)A\int_{B_r\setminus B_{r-T}} (W(\sigma) - W(u))dx + \frac{C(n)}{A}\int_{(B_r\setminus B_{r-T})\cap\{q^u-q^\sigma\le\lambda\}} (q^u - q^\sigma)^2 dx$$

$$I_2 = -2C(n)A\int_{B_r\setminus B_{r-T}} \nabla q^\sigma(\nabla q^u - \nabla q^\sigma)dx$$

Assume $\lambda \le \min\{\rho_0, 1\}$. Then there exists a constant $\tilde{C} > 0$ independent of r, such that

$$I_1 \le \tilde{C}A\mathcal{L}^n((B_r \setminus B_{r-T}) \cap \{q^u > \lambda\}) + \frac{\tilde{C}}{A}\int_{(B_r\setminus B_{r-T})\cap\{q^u\le\lambda\}} W(u)dx \tag{85}$$

$$A > 0$$

To see this we proceed by splitting the integration over $B_r \setminus B_{r-T}$ into integrations over $\{q^u \leq \lambda\}$ and $\{q^u > \lambda\}$

From $q^\sigma \leq q^u, q^u \leq \lambda \leq \rho_0$ by the monotonicity of W near $u = a$

$$\int_{B_r \setminus B_{r-T}} (W(\sigma) - W(u))dx \leq dx \leq 0$$

Thus

$$\int_{B_r \setminus B_{r-T}} (W(\sigma) - W(u))dx \leq \mathcal{W}_M \mathcal{L}^n((B_r \setminus B_{r-T}) \cap \{q^u > \lambda\})$$

For the 2$^{\text{nd}}$ term we can utilize the lower bound in $(H_d)_{(i)}$:

In $q^\sigma \leq q^u \leq \lambda \leq min\{\rho_0, 1\}$

$$W(u) \geq C^*(q^u)^\alpha \geq C^*(q^u - q^\sigma)^\alpha \geq C^*(q^u - q^\sigma)^2$$

hence

$$\int_{(B_r \setminus B_{r-T}) \cap \{q^u \leq \lambda\}} (q^u - q^\sigma)^2 dx \leq \frac{1}{C^*} \int_{(B_r \setminus B_{r-T}) \cap \{q^u \leq \lambda\}} \mathcal{W}(u)dx$$

and (84) is established with $\tilde{C} = C(n)max\{\frac{1}{C^*}, 2\mathcal{W}_M\}$.

Next we take up $I_2, \lambda \leq min\{\rho_0, 1\}$.

We will show that there exists $\hat{C} > 0$, independent of r, but depending on M , T, such that:

$$I_2 \leq \hat{C}A\mathcal{L}^n((B_r \setminus B_{r-T}) \cap \{q^u > \lambda\}) + \hat{C}A \int_{(B_r \setminus B_{r-T}) \cap \{q^u \leq \lambda\}} W(u)dx \tag{86}$$

We proceed as follows. Let $q^h(x)$ as in Exercise 12

$$I_2 = -2C(n)A \int_{(B_r \setminus B_{r-T}) \cap \{q^h < q^u\}} \nabla q^\sigma (\nabla q^u - \nabla q^\sigma)dx - 2C(n)A \int_{(B_r \setminus B_{r-T}) \cap \{q^h \leq q^u\}} \nabla q^\sigma (\nabla q^u - \nabla q^\sigma)dx$$

(trivially, since $q^\sigma = min\{q^h, q^u\}$)

$$= -2C(n)A \int_{(B_r \setminus B_{r-T}) \cap \{q^h < q^u\}} \nabla q^h (\nabla q^u - \nabla q^h)dx = 2C(n)A \int_{(B_r \setminus B_{r-T}) \cap \{q^h < q^u\}} \Delta q^h (q^u - q^h)dx$$

We now split the integral $I_2 = I_2^+ + I_2^-$ where I_2^+, I_2^- correspond to the integration over

$\{q^u > \lambda\}$ and $\{q^u \le \lambda\}$.

Then we have the estimates

$$I_2^+ \le 2CAC_M M\mathcal{L}^n((B_r \setminus B_{r-T}) \cap \{q^u - \lambda\})$$
$$I_2^- \le 2CAC_1 \int_{(B_r \setminus B_{r-T}) \cap \{q^h < q^u\} \cap \{q^u \le \lambda\}} (q^h)^{\tau-1}(q^u - q^h)dx$$

In the 1st we take $\Delta q^h \le C_M$. In the 2nd we utilize $\Delta q^h \le C_1(q^h)^{\tau-1}$, C_M and C_1 constants bounded as as $\alpha \to 0$. Moreover in the 2nd term we have:

$$(q^h)^{\tau-1}(q^u - q^h) \le (q^u)^{\tau-1}(q^u - q^h) \le (q^u)^\alpha \le \frac{1}{C^*}W(u)$$

and so (86) is established.

Recalling (63) and collecting all the estimates above, we have for fixed $A > A_0$:

$$\lambda^2(V(r-T))^{\frac{n-1}{n}} + CA\,A(r-T)$$
$$\le (\tilde{C} + \hat{C})A(V(r) - V(r-T)) + (\frac{\tilde{C}}{A} + \hat{C}A)(A(r) - A(r-T))$$

and rearranging

$$C(\lambda)\left((V(r-T))^{\frac{n-1}{n}} + A(r-T)\right) \le (V(r) - V(r-T)) + (A(r) - A(r-T)) \tag{87}$$

where

$$C(\lambda) = \frac{min\{\lambda^2, CA\}}{max\{(\tilde{C} + \hat{C})A, \frac{\hat{C}}{A} + \hat{C}A\}}$$

From this difference scheme we obtain (Exercise 12)

$$V(kT) + A(kT) \ge Ck^n, \quad \text{for } k \ge k_0, \quad k \text{ integer} \tag{88}$$

utilizing

$$A(kT) \le \hat{C}_0(kT)^{n-1} \quad \text{(by Lemma 3.4)} \tag{89}$$

we conclude the proof of Theorem 3.5 for $0 < a < 2$. □

Remark 3.6.
1) The case $\alpha = 2$ is more involved since there is no comparison function q^h that vanishes on ∂B_{r-T}. Instead one has to resort to the linear equation

$$\begin{cases} \Delta\phi = c_0\phi \;\; in \;\; B_r \\ \phi = 1 \end{cases}$$

and construct a comparison function that is exponentially small on B_{r-T}, T large,

$$q^h \leq Me^{-C_1 T} \;\; on \; B_{r-T}$$

We refer to [1].
2) For $0 < \alpha < 1$ the L^∞ gradient bound is not appropriate since $u \in C^{\beta}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ for some $\beta \in (0, 1)$. The only prerequisite for the proof on the case $\alpha \in (0, 2)$ is Lemma 3.4 that can be established with a different proof ([1]).

References

[1] Nicholas D. Alikakos, Giorgio Fusco, Panayotis Smyrnelis: *Elliptic Systems of Phase Transition Type* , Progress in Nonlinear Differential Equations and Their Applications (2018), Birkhauser